MID-SEMESTER EXAMINATION
M. MATH II YEAR, II SEMESTER February 17, 2017
Operator Theory

1. Let $\mathcal{H}$ be a Hilbert space of dimesion greater than 1. Show that the space $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is not Hilbert space.

Solution: Every Hilbert space satifies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

for all $x, y \in \mathcal{H}$. Since the space has dimension more than one, we can choose an orthonormal set $\{u, v\}$. Consider the diagonal (one rank) operators A and B given by them:

$$A(x) = \langle x, u \rangle u, \quad B(x) = \langle x, v \rangle v,$$

where the bra-ket is the inner product in $\mathcal{H}$. We can see that these operators do not satisfy the above identity in $\mathcal{B}(\mathcal{H})$ with respect to the operator norm. It is easy to see that

$$\|A\| = 1 = \|B\|, \quad \|A+B\| = 1 = \|A-B\|.$$

2. Let N be a normal operator on a Hilbert space $\mathcal{H}$. Show that if λ is an eigenvalue of N, then its conjugate is an eigenvalue of N^*. Show that eigenvectors corresponding to distinct eigenvalues are orthogonal.

Solution: This is standard: It is enough to prove that the Kernel of N and that of N^* are the same, which is proved through Ker$N$=KerN^*N. Observe also that $N - \lambda I$ is normal all λ. See Rudin or Kreyzing.

3. Show that the spectrum of an orthogonal projection P on a Hilbert space is contained in $\{0, 1\}$.

Solution: If $\alpha \neq 0, 1$, then $\alpha - P$ is inveritble and the inverse is

$$\frac{I}{\alpha} + \frac{P}{\alpha(1-\alpha)},$$

which is a direct verification.

4. Show that the unilateral forward shift on ℓ^2 is not compact.

Solution: Standard. It enough to prove the existence of a bounded set whose image under the shift is unbounded. Just consider the standard orthonormal basis in the space.

5. Let $\mathcal{A}$ be a commutative unital Banach algebra and let $\hat{\mathcal{A}}$ be its Gelfand dual. Show that maximal ideals of $\mathcal{A}$ are in one-one correspondence with $\hat{\mathcal{A}}$.

Solution: Standard.

6. Let $\mathcal{A}$ be a unital Banach algebra and let $\rho(x)$ be the resolvent of $x \in \mathcal{A}$. If φ is a continuous linear functional, show that the map

$$(\varphi(x-z)^{-1})$$

defined on $\rho(x)$ is a complex differentiable function.

Solution: Standard.

7. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a unital $*$-homomorphism. Show that $\sigma(\pi(a)) \subseteq \sigma(a)$. Using this or otherwise, show that $\|\pi(a) \leq \|a\|$ for all $a \in \mathcal{A}$.

Solution: See any book on C^*-algebra.